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On fluid motions induced by an electromagnetic field in a liquid drop immersed in a conducting fluid

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The deformation of a liquid drop immersed in a conducting fluid by the imposition of a uniform electric field is investigated. The flow field set up is due to the surface charge and the tangential electric field stress over the surface of the drop, and the rotationality of the Lorentz force which is set up by the electric current and the associated magnetic field. It is shown that when the fluids are poor conductors and good dielectrics the effects of the Lorentz force are minimal and the flow field is due to the stresses of the electric field tangential to the surface of the drop, in agreement with other authors. When, however, the fluids are highly conducting and poor dielectrics the effects of the Lorentz force may be predominant, especially for larger drops.

1. Introduction

A liquid drop immersed in a fluid which is subjected to a uniform electric field becomes prolate or oblate and is very nearly a spheroid, with the axis of the spheroid in the direction of the impressed field (O'Konski & Thacher 1953; O'Konski & Harris 1957; Allan & Mason 1962; Garton & Krasucki 1964; Taylor 1966; Torza, Cox & Mason 1971), and usually bursts at high fields. The deformation is due to the stress exerted by electric field over the surface of the drop. When the medium is perfect dielectric, that is, when the electric conductivity σ of the medium is zero, the drop always takes a prolate shape before bursting. This is explained in terms of the surface tension and the electric field stress, which is normal to the surface of the drop (Garton & Krasucki 1964).

When the fluid is slightly conducting it was observed that the drop sometimes takes an oblate shape (Allan & Mason 1962). Taylor (1966) pointed out that in this case there is a surface charge over the interface of the drop and external fluid, and consequently the electric field stress has a component tangential to the surface of the drop. This can be balanced by hydrodynamic stresses produced by liquid flow in the drop and its surroundings. Taylor proposed an electrohydrodynamic theory to explain this. The theory is in reasonably good agreement with observation [see addendum to Taylor's (1966) paper by McEwan & DeLong and Torza *et al.* (1971)].

The flow field of Taylor's solution exterior to the drop is very similar to that produced by the passage of a uniform electric current parallel to the axis of an axisymmetric solid body immersed in a fluid of different electrical conductivity

(Chow & Halat 1969; Sozou 1970*a*, *b*). In this case the flow field is produced by the rotational Lorentz force due to the distorted electric current and the associated magnetic field. Since here we are concerned with slow (Stokes) flows, and the components of the stress of both the electrohydrodynamic and magnetohydrodynamic flow fields normal and tangential to the surface of the drop have the same angular dependence, we can superimpose these fields. This rearranges but does not upset the balance of stresses over the surface of the drop. Below, we present a complete theory which takes account of both the magnetic and electric effects. For fluids which are poor conductors the magnetic effects are negligible and the theory, in effect, reduces to Taylor's work. For fluids which are good conductors and poor dielectrics, especially for larger drops, the magnetohydrodynamic effects may be dominant. The distortion of the drop and the flow field set up may entirely be due to the rotationality of the Lorentz force. When the drop is replaced by a solid body only the magnetohydrodynamically induced component of the external flow field remains.

2. Electromagnetic equations and stresses

We consider an infinite incompressible viscous conducting fluid containing an immiscible drop, assumed spherical, of another fluid. We use a spherical polar co-ordinate system (r, θ, ϕ), with the origin at the centre of the drop, and assume that a uniform electrostatic field $\mathbf{E}$ is imposed parallel to the direction $\theta = 0$. Let σ denote the electrical conductivity, κ the dielectric constant, μ the coefficient of viscosity and χ the permeability of the fluid. The electric current density is $\mathbf{J}$, the magnetic field is $\mathbf{H}$ and the magnetic induction is $\mathbf{B}$. Let the suffix 1 refer to the fluid extending to infinity and the suffix 2 to the liquid drop. Our steady-state electromagnetic variables, in the absence of fluid motions, can be obtained from books on electromagnetism and in rationalized mks units are given by

$$\mathbf{J}_1 = \sigma_1 \mathbf{E}_1 = \left[\sigma_1 E\left(1 - 2\frac{1-R}{2+R}\frac{a^3}{r^3}\right)\cos\theta,\ -\sigma_1 E\left(1+\frac{1-R}{2+R}\frac{a^3}{r^3}\right)\sin\theta, 0\right], \tag{1}$$

$$\mathbf{J}_2 = \sigma_2 \mathbf{E}_2 = \left(\frac{3\sigma_2 E}{2+R}\cos\theta,\ -\frac{3\sigma_2 E}{2+R}\sin\theta, 0\right), \tag{2}$$

$$\mathbf{B}_1 = \chi_1 \mathbf{H}_1 = \hat{\boldsymbol{\phi}} \times \tfrac{1}{2}\chi_1\sigma_1 E\left(r - 2\frac{1-R}{2+R}\frac{a^3}{r^2}\right)\sin\theta, \tag{3}$$

$$\mathbf{B}_2 = \chi_2 \mathbf{H}_2 = \hat{\boldsymbol{\phi}} \times \frac{3\chi_2\sigma_2 E}{2(2+R)} r\sin\theta, \tag{4}$$

$$\epsilon_0\kappa_1 E_{1r} - \epsilon_0\kappa_2 E_{2r} = -\rho \quad \text{at} \quad r = a, \tag{5}$$

where a is the radius of the drop, $R = \sigma_2/\sigma_1$, ϵ_0 the permitivity of free space, E_{1r} and E_{2r} the radial components of $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively, and ρ the surface charge at the interface between the drop and the rest of the fluid.

The radial, p_{rr}, and tangential, $p_{r\theta}$, components of the stress exerted on the surface of the drop by the electric and magnetic fields are given by

$$(p_{rr})_E = \epsilon_0(-\tfrac{1}{2}\kappa_1 E_1^2+\kappa_1 E_{1r}^2+\tfrac{1}{2}\kappa_2 E_2^2-\kappa_2 E_{2r}^2)$$

$$= [9\epsilon_0 E^2/2(2+R)^2]\,[\kappa_2-\kappa_1+\{\kappa_1(R^2+1)-2\kappa_2\}\cos^2\theta], \tag{6}$$

$$(p_{r\theta})_E = \epsilon_0(\kappa_1 E_{1r}E_{1\theta}-\kappa_2 E_{2r}E_{2\theta}) = -(9\epsilon_0 E^2/(2+R)^2)\,(\kappa_1 R-\kappa_2)\sin\theta\cos\theta, \tag{7}$$

$$(p_{rr})_B = -\tfrac{1}{2}\chi_1 H_1^2+\tfrac{1}{2}\chi_2 H_2^2 = \tfrac{1}{8}\,(3R\sigma_1 E/(2+R))^2\,(\chi_2-\chi_1)\,a^2\,(1-\cos^2\theta), \tag{8}$$

$$(p_{r\theta})_B = 0, \tag{9}$$

where $E_{1\theta}$ and $E_{2\theta}$ are the θ components of $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively.

3. Fluid motions and stresses

As pointed out by Taylor (1966), for equilibrium, the stress $(p_{r\theta})_E$ at the surface of the drop must be balanced and this can only be done by the viscous stresses associated with a flow field in the drop and its surroundings. The flow field for the case when the stresses due to the inertia are negligible in comparison with the ones due to viscosity (Stokes flow) was discussed by Taylor (1966). The flow field, which is in reasonable agreement with observation, is symmetrical about a plane containing the electric field and also about a plane through the centre of the drop perpendicular to the electric field. Thus the flow field exterior to the drop is similar to that produced by the $\mathbf{J}\times\mathbf{B}$ force associated with the passage of a uniform electric current parallel to the axis of an axisymmetric solid body immersed in a fluid of different electrical conductivity (Chow & Halat 1969; Sozou 1970*a*, *b*). The two flow fields may therefore coexist, or the flow field which is exterior to the drop and was discussed by Taylor may be produced by the $\mathbf{J}\times\mathbf{B}$ force.

Following Taylor (1966) we assume that the velocity is small, ignore the convection of the surface charge ρ by the hydrodynamic currents and assume that in the momentum equation the inertia terms are negligible in comparison with the viscous ones. We also ignore the effect of the velocity on the electromagnetic field, that is, we assume that the flow field does not affect the electromagnetic variables, which are still given by (1)–(5).

In terms of the stream function ψ the fluid velocity $\mathbf{v}$ is given by

$$\mathbf{v} = \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},0\right). \tag{10}$$

The momentum equation is

$$\nabla p = \mathbf{J}\times\mathbf{B}+\mu\nabla^2\mathbf{v}, \tag{11}$$

where p is the fluid pressure. When we make use of (1)–(4) and (10), and take the curl of (11) we obtain

$$D^4\psi_1 = -\frac{3\chi_1(1-R)\,\sigma_1^2 E^2}{\mu_1(2+R)}\frac{a^3}{r^2}\left(1-2\frac{1-R}{2+R}\frac{a^3}{r^3}\right)\sin^2\theta\cos\theta, \tag{12}$$

$$D^4\psi_2 = 0, \tag{13}$$

where
$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{14}$$

The solution of (12) and (13) that would produce the appropriate stress to balance the $(p_{r\theta})_E$ at the surface of the drop (Taylor 1966) and also have the geometry due to the symmetry of the $\mathbf{J}\times\mathbf{B}$ force about the planes $\theta = \pm\frac{1}{2}\pi$ and $\theta = 0$, $\theta = \pi$ is

$$\psi_1 = \left[A_1 a^4 r^{-2} + C_1 a^2 - \lambda\frac{1-R}{2+R}\left(r^2 + 2\frac{1-R}{2+R}a^3 r^{-1}\right)\right]\sin^2\theta\cos\theta, \tag{15}$$

$$\psi_2 = (A_2 a^{-1} r^3 + C_2 a^{-3} r^5)\sin^2\theta\cos\theta, \tag{16}$$

where the A's and the C's are constants and $\lambda = \chi_1\sigma_1^2 E^2 a^3/8\mu_1$. The velocity field represented by (15) is finite, instead of zero, at infinity. This is due to the neglect of the inertia terms from the momentum equation (Sozou 1970*a*).

As always, p is obtained by integrating the momentum equation. After a little algebra it is found that the hydrodynamic stresses,

$$(p_{rr})_H = -p + 2\mu\frac{\partial}{\partial r}(\mathbf{v}.\hat{\mathbf{r}}), \quad (p_{r\theta})_H = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\mathbf{v}.\hat{\boldsymbol{\theta}}}{r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}(\mathbf{v}.\hat{\mathbf{r}})\right],$$

at the surface of the drop are given by

$$a\mu_1^{-1}(p_{rr})_{1H} = -\left[8A_1 + 6C_1 - 2\lambda\frac{8-16R+5R^2}{(2+R)^2}\right] \times (3\cos^2\theta - 1) + \frac{12\lambda}{(2+R)^2} + \text{constant}, \tag{17}$$

$$a\mu_1^{-1}(p_{r\theta})_{1H} = -\left[16A_1 + 6C_1 - 4\lambda\frac{7-11R+4R^2}{(2+R)^2}\right]\sin\theta\cos\theta, \tag{18}$$

$$a\mu_2^{-1}(p_{rr})_{2H} = (2A_2 - C_2)(3\cos^2\theta - 1) + \tfrac{1}{4}a^3\chi_2\mu_2^{-1}\left(\frac{3R\sigma_1 E}{2+R}\right)^2 \times (1-\cos^2\theta) + \text{constant}, \tag{19}$$

$$a\mu_2^{-1}(p_{r\theta})_{2H} = -(6A_2 + 16C_2)\sin\theta\cos\theta. \tag{20}$$

At $r = a$ the radial component of the velocity is zero and the tangential component is continuous, that is,

$$A_1 + C_1 - \lambda(4-5R+R^2)/(2+R)^2 = 0 = A_2 + C_2,$$

and
$$-2A_1 - 2\lambda(1+R-2R^2)/(2+R)^2 = 3A_2 + 5C_2,$$

so that
$$C_1 = -A_1 + \lambda(4-5R+R^2)/(2+R)^2, \tag{21}$$

$$A_2 = A_1 + \lambda(1+R-2R^2)/(2+R)^2, \tag{22}$$

$$C_2 = -A_1 - \lambda(1+R-2R^2)/(2+R)^2. \tag{23}$$

At $r = a$ the stresses must balance, i.e.

$$(p_{rr})_B + (p_{rr})_E + (p_{rr})_{1H} - (p_{rr})_{2H} = 2T/a, \tag{24}$$

$$(p_{r\theta})_E + (p_{r\theta})_{1H} - (p_{r\theta})_{2H} = 0, \tag{25}$$

or

$$-9\lambda\mu_1 R^2(1+\chi_2/\chi_1)(1-\cos^2\theta)+\tfrac{9}{2}\epsilon_0 E^2 a[\kappa_2-\kappa_1+\{\kappa_1(1+R^2)-2\kappa_2\}\cos^2\theta]$$
$$+12\lambda\mu_1-[(2\mu_1+3\mu_2)(2+R)^2 A_1+\lambda\mu_1(8+2R-4R^2)+\lambda\mu_2(3+3R-6R^2)]$$
$$\times(3\cos^2\theta-1) = 2T(2+R)^2+\text{constant}, \quad (26)$$

$$[10(\mu_1+\mu_2)(2+R)^2 A_1+9\epsilon_0 aE^2(\kappa_1 R-\kappa_2)-\lambda\mu_1(4-14R+10R^2)$$
$$+10\lambda\mu_2(1+R-2R^2)]\sin\theta\cos\theta = 0, \quad (27)$$

where T is the surface tension acting over the spherical drop. If we equate to zero the coefficient of $\sin\theta\cos\theta$ in (27) and the coefficients of $\cos^2\theta$ in (26), after a little algebra we obtain

$$A_1 = -\frac{\lambda}{10(1+M)(2+R)^2}[10+10R-20R^2-M(4-14R+10R^2)+M\lambda_0(SR-1)], \quad (28)$$

$$S(1+R^2)-2+\frac{3}{5}\frac{3+2M}{1+M}(SR-1)$$
$$= \frac{6}{\lambda_0}\left[\frac{7-2R-5R^2}{5(1+M)}+\frac{44-4R-(25+15/Q)R^2}{5}\right], \quad (29)$$

where $M = \mu_1/\mu_2$, $S = \kappa_1/\kappa_2$, $Q = \chi_1/\chi_2$ and $\lambda_0 = 72\epsilon_0\kappa_2/(\chi_1\sigma_1^2 a^2)$. In practice $Q \doteqdot 1$.

4. Discussion

When (29) is satisfied the liquid drop has a spherical shape. If (29) is not satisfied the drop will become oblate or prolate. In order to find out whether under conditions where (29) is not quite satisfied the drop will become oblate or prolate we employ Taylor's (1966) technique and assume that the application of a stress $C\cos^2\theta$ normally inward to the surface of the drop is necessary to keep the drop spherical. If C is positive, that is if at the poles ($\theta = 0, \theta = \pi$) the stress has to be applied in the inward direction to keep the drop spherical, the drop must be elongated in the direction of the field. If C is negative the drop is oblate. If we replace T on the right-hand side of (26) by $T+C\cos^2\theta$ and, making use of (28), equate the coefficients of $\cos^2\theta$ on the two sides of the resulting equation we find

$$C = \frac{9\epsilon_0 a\kappa_2 E^2}{4(2+R)^2}\left[S(1+R^2)-2+\frac{3}{5}\frac{3+2M}{1+M}(SR-1)\right.$$
$$\left.-\frac{6}{\lambda_0}\left\{\frac{7-2R-5R^2}{5(1+M)}+\frac{44-4R-(25+15/Q)R^2}{5}\right\}\right]. \quad (30)$$

Equations (29) and (30) are generalizations of the expressions derived by Taylor [his equations (24) and (25), corrected for an arithmetic error in deriving the normal hydrodynamic stress corresponding to the second term of ψ_2] for the case when the magnetic effects are neglected. They reduce to Taylor's expressions when the right-hand side of (29) is set equal to zero. Taylor's expressions were also derived by Torza *et al.* (1971) as a special case of the problem where the electric field varies harmonically with respect to time. Indeed, Torza *et al.* attempted an estimate of the deviation of the shape of the drop from spherical. They assumed that for small deformations the drop is a spheroid – prolate or oblate – of small eccentricity, that is, they assumed that the surface of the drop is given by

$$r = a[1+\epsilon(3\cos^2\theta - 1)],$$

where ϵ is small. The discontinuity of normal stress due to the surface tension is now given by $T(r_1^{-1}+r_2^{-1})$, where r_1 and r_2 are the principal radii of curvature at any point of the drop surface, or

$$\tfrac{1}{2}aT(r_1^{-1}+r_2^{-1}) = T(1-2\epsilon+6\epsilon\cos^2\theta). \tag{31}$$

If we now replace T on the right-hand side of (26) by the right-hand side of (31) and then equate the coefficients of $\cos^2\theta$ on the two sides of the resulting equation we find that

$$\epsilon = C/(6T).$$

The sign of ϵ, like the sign of C in (30), shows whether the deformed surface is oblate or prolate. The experimental results of Torza *et al.* (1971) are in qualitative agreement with the theory, though the deformations were found to be larger than the theoretical predictions. Note also that the approximation of the deformed surface by a spheroid is not necessarily accurate. For an accurate approximation we must, of course, ensure that over the surface of the drop there is balance, correct to order ϵ, in the tangential and normal stress. This requires modifications, correct to order ϵ, of the velocity and electromagnetic field and thus of (27) and *both* sides of (26).

The liquids used by Allan & Mason (1962) and Torza *et al.* (1971) are poor conductors and make λ_0 very large and the magnetic effects much smaller than the ones due to the electric field. For these liquids Taylor's suggestion, that the observed flow patterns are due to the stress of the electric field, is correct. When, however, the liquids are poor dielectrics and good conductors, especially for larger drops (that is, when λ_0 is not large) the magnetic effects due to the electric current become important and in the case where $SR \doteqdot 1$ they must be the dominant ones. In the special case when the fluid external to the drop is highly (almost perfectly) conducting λ_0 is very small, $R \doteqdot 0$ and, provided $S < \frac{16}{5}$, C in (30) is negative, i.e. the drop will become oblate. A_2 is positive and at the interface of the drop the direction of circulation is from pole ($\theta = 0$, and $\theta = \pi$) to equator ($\theta = \pm\frac{1}{2}\pi$) [Taylor 1966, figure 1].

Equations (22), (23) and (28) show that when account is taken of the magnetic effects of the electric currents it is possible, if there are liquids with the appropriate properties, to find configurations where $A_2 = C_2 = 0$, that is, the tangential stress of the electric field at the interface is balanced by the stresses of the external flow field there, and the flow field inside the drop is zero. It would be of interest to subject bubbles immersed in highly conducting fluids to the influence of a uniform electrical field (or electric current) and test the theory presented here experimentally.

Finally we note Taylor's remark that if the drop is replaced by a solid body no flow is induced in the fluid by the tangential component of the electric field. In such a case, however, flow is induced by the Lorentz force. Since for threedimensional configurations this force is rotational (Sozou 1970*a*) it cannot be balanced by the hydrostatic fluid pressure and sets the fluid in motion. Thus for a solid sphere we set $A_2 = 0$, that is,

$$A_1 = -\lambda(1+R-2R^2)^2/(2+R)^2, \quad C_1 = \lambda(5-4R-R^2)/(2+R)^2,$$

and obtain the case discussed by Chow & Halat (1969).

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